**Sage Quick Reference:**
**Elementary Number Theory**
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Sage Version 3.4
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以下 $m, n, a, b, \ldots$ は `ZZ` の元とする.
`ZZ` $= \mathbb{Z} =$ 全ての整数
Everywhere $m, n, a, b, etc.$ are elements of `ZZ`
`ZZ` $= \mathbb{Z} =$ all integers

## 整数 Integers

$\ldots, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$

$n$ を $m$ で割ると余りは `n % m`
`gcd(n,m)`, `gcd(`*list*`)`
拡張された公約数 $g = sa + tb = \gcd(a, b)$: `g,s,t=xgcd(a,b)`
`lcm(n,m)`, `lcm(`*list*`)`
二項係数 $\binom{m}{n}$ = `binomial(m,n)`
*base* 進法による表示: `n.digits(`*base*`)`
*base* 進法による桁数: `n.ndigits(`*base*`)`
(*base* は省略可, デフォルトは 10)
割り切る. $n \mid m$: `n.divides(m)`, $nk = m$ を満たす $k$ があるか.
約数 – $d \mid n$ を満たす $d$ 達 : `n.divisors()`
階乗 – $n!$ = `n.factorial()`

$n$ divided by $m$ has *remainder* `n % m`
`gcd(n,m)`, `gcd(`*list*`)`
extended gcd $g = sa + tb = \gcd(a, b)$: `g,s,t=xgcd(a,b)`
`lcm(n,m)`, `lcm(`*list*`)`
binomial coefficient $\binom{m}{n}$ = `binomial(m,n)`
digits in a given base: `n.digits(`*base*`)`
number of digits: `n.ndigits(`*base*`)`
(*base* is optional and defaults to 10)
divides $n \mid m$: `n.divides(m)` if $nk = m$ some $k$
divisors – all $d$ with $d \mid n$: `n.divisors()`
factorial – $n!$ = `n.factorial()`

## 素数 Prime Numbers

$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \ldots$

素因数分解: `factor(n)`
素数判定: `is_prime(n)`, `is_pseudoprime(n)`
素冪判定: `is_prime_power(n)`
$\pi(x) = \#\{p : p \le x \text{ is prime}\}$ = `prime_pi(x)`
素数の集合: `Primes()`
$\{p : m \le p < n \text{ and } p \text{ prime}\}$ =`prime_range(m,n)`
n 以上 m 以下の素冪の集合: `prime_powers(m,n)`
最初の $n$ 個の素数: `primes_first_n(n)`
次の素数, ひとつ前の素数: `next_prime(n)`, `previous_prime(n)`, `next_probable_prime(n)`
次の素冪, ひとつ前の素冪: `next_prime_power(n)`, `pevious_prime_power(n)`
$2^p - 1$ の素数性に関する Lucas-Lehmer テスト

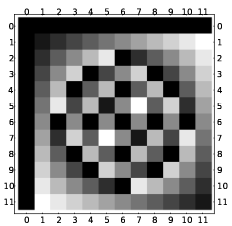
```
def is_prime_lucas_lehmer(p):
    s = Mod(4, 2^p - 1)
    for i in range(3, p+1): s = s^2 - 2
    return s == 0
```

factorization: `factor(n)`
primality testing: `is_prime(n)`, `is_pseudoprime(n)`
prime power testing: `is_prime_power(n)`
$\pi(x) = \#\{p : p \le x \text{ is prime}\}$ = `prime_pi(x)`
set of prime numbers: `Primes()`
$\{p : m \le p < n \text{ and } p \text{ prime}\}$ =`prime_range(m,n)`
prime powers: `prime_powers(m,n)`
first $n$ primes: `primes_first_n(n)`
next and previous primes: `next_prime(n)`, `previous_prime(n)`, `next_probable_prime(n)`
prime powers: `next_prime_power(n)`, `pevious_prime_power(n)`
Lucas-Lehmer test for primality of $2^p - 1$

```
def is_prime_lucas_lehmer(p):
    s = Mod(4, 2^p - 1)
    for i in range(3, p+1): s = s^2 - 2
    return s == 0
```

## 合同式, モジュラ計算 Modular Arithmetic and Congruences

`k=12; m = matrix(ZZ, k, [(i*j)%k for i in [0..k-1] for j in [0..k-1]]); m.plot(cmap='gray')`

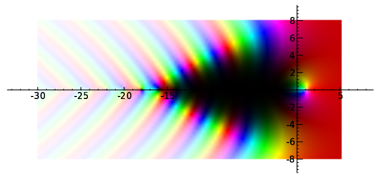


オイラーの $\phi(n)$ 関数: `euler_phi(n)`
クロネッカーシンボル $\left(\frac{a}{b}\right)$ = `kronecker_symbol(a,b)`
平方剰余: `quadratic_residues(n)`
平方非剰余: `quadratic_residues(n)`
環 $\mathbb{Z}/n\mathbb{Z}$ = `Zmod(n) = IntegerModRing(n)`
$\mathbb{Z}/n\mathbb{Z}$ の元としての $a$ ($a \bmod n$): `Mod(a, n)`
$\mathbb{Z}/n\mathbb{Z}$ での原始根 = `primitive_root(n)`
$\mathbb{Z}/n\mathbb{Z}$ での逆元: `n.inverse_mod(m)`
$\mathbb{Z}/n\mathbb{Z}$ での冪 $a^n \pmod m$: `power_mod(a, n, m)`
中国の剰余定理: `x = crt(a,b,m,n)`
$x \equiv a \pmod m$ かつ $x \equiv b \pmod n$ を満たす $x$ を探す
離散対数: `log(Mod(6,7), Mod(3,7))`
$a \pmod n$ の次数 = `Mod(a,n).multiplicative_order()`
$a \pmod n$ の平方根 = `Mod(a,n).sqrt()`

Euler's $\phi(n)$ function: `euler_phi(n)`
Kronecker symbol $\left(\frac{a}{b}\right)$ = `kronecker_symbol(a,b)`
Quadratic residues: `quadratic_residues(n)`
Quadratic non-residues: `quadratic_residues(n)`
ring $\mathbb{Z}/n\mathbb{Z}$ = `Zmod(n) = IntegerModRing(n)`
$a$ modulo $n$ as element of $\mathbb{Z}/n\mathbb{Z}$: `Mod(a, n)`
primitive root modulo $n$ = `primitive_root(n)`
inverse of $n \pmod m$: `n.inverse_mod(m)`
power $a^n \pmod m$: `power_mod(a, n, m)`
Chinese remainder theorem: `x = crt(a,b,m,n)`
finds $x$ with $x \equiv a \pmod m$ and $x \equiv b \pmod n$
discrete log: `log(Mod(6,7), Mod(3,7))`
order of $a \pmod n$ = `Mod(a,n).multiplicative_order()`
square root of $a \pmod n$ = `Mod(a,n).sqrt()`

## 特殊関数 Special Functions

`complex_plot(zeta, (-30,5), (-8,8))`



$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} = \sum \frac{1}{n^s} = \texttt{zeta(s)}$$

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\log(t)} dt = \texttt{Li(x)}$$

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt = \texttt{gamma(s)}$$

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## 連分数 Continued Fractions

`continued_fraction(pi)`

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}}$$

連分数: `c=continued_fraction(x,`*bits*`)`
近似分数 (達): `c.convergents()`
部分分子 $p_n$ = `c.pn(n)`
部分分母 $q_n$ = `c.qn(n)`

値: `c.value()`

continued fraction: `c=continued_fraction(x,` *bits*`)`
convergents: `c.convergents()`
convergent numerator $p_n$ = `c.pn(n)`
convergent denominator $q_n$ = `c.qn(n)`
value: `c.value()`

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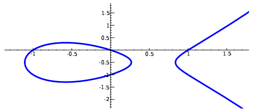
## 楕円曲線　Elliptic Curves

`EllipticCurve([0,0,1,-1,0]).plot(plot_points=300,thickness=3)`



`E = EllipticCurve(`$[a_1, a_2, a_3, a_4, a_6]$`)`

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

$E$ の導手 (conductor) $N$ = `E.conductor()`
$E$ の判別式 $\Delta$ =`E.discriminant()`
$E$ の階数 = `E.rank()`
$E(\mathbb{Q})$ の自由生成系 = `E.gens()`
$j$-invariant = `E.j_invariant()`
$N_p = \#\{$modulo $p$ での $E$ の解 $\}$ = `E.Np(`*prime*`)`
$a_p = p + 1 - N_p$ =`E.ap(`*prime*`)`
$L(E,s) = \sum \frac{a_n}{n^s}$ = `E.lseries()`
$\mathrm{ord}_{s=1} L(E,s)$ = `E.analytic_rank()`

`E = EllipticCurve(`$[a_1, a_2, a_3, a_4, a_6]$`)`

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

conductor $N$ of $E$ =`E.conductor()`
discriminant $\Delta$ of $E$ =`E.discriminant()`
rank of $E$ = `E.rank()`
free generators for $E(\mathbb{Q})$ = `E.gens()`
$j$-invariant = `E.j_invariant()`
$N_p = \#\{$solutions to $E$ modulo $p\}$ = `E.Np(`*prime*`)`
$a_p = p + 1 - N_p$ =`E.ap(`*prime*`)`
$L(E,s) = \sum \frac{a_n}{n^s}$ = `E.lseries()`
$\mathrm{ord}_{s=1} L(E,s)$ = `E.analytic_rank()`

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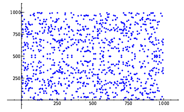
## $p$ で合同な楕円曲線　Elliptic Curves Modulo $p$

`EllipticCurve(GF(997), [0,0,1,-1,0]).plot()`



`E = EllipticCurve(GF(p),` $[a_1, a_2, a_3, a_4, a_6]$`)`
$\#E(\mathbb{F}_p)$ = `E.cardinality()`
$E(\mathbb{F}_p)$ の生成系 = `E.gens()`
$E(\mathbb{F}_p)$ =`E.points()`

`E = EllipticCurve(GF(p),` $[a_1, a_2, a_3, a_4, a_6]$`)`
$\#E(\mathbb{F}_p)$ = `E.cardinality()`
generators for $E(\mathbb{F}_p)$ = `E.gens()`
$E(\mathbb{F}_p)$ =`E.points()`